Extended Painlevé Expansion, Nonstandard Truncation and Special Reductions of Nonlinear Evolution Equations

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To study a nonlinear partial differential equation (PDE), the Painlevé expansion developed by Weiss, Tabor and Carnevale (WTC) is one of the most powerful methods. In this paper, using any singular manifold, the expansion series in the usual Painlevé analysis is shown to be resummable in some different ways. A simple nonstandard truncated expansion with a quite universal reduction function is used for many nonlinear integrable and nonintegrable PDEs such as the Burgers, Korteweg de-Vries (KdV), Kadomtsev-Petviashvli (KP), Caudrey-Dodd-Gibbon-Sawada-Kortera (CDGSK), Nonlinear Schrödinger (NLS), Davey-Stewartson (DS), Broer-Kaup (BK), KdV-Burgers (KdVB), $\lambda\phi^4$, sine-Gordon (sG) etc.

1. Introduction

There are some marvellous methods to study the integrability of a nonlinear partial differential equation (PDE). The Painlevé analysis developed by Weiss, Tabor and Carnevale (WTC) [1] is one of the most effective approaches. Applying the WTC approach to nonlinear PDEs, one can obtain not only properties like the Painlevé property, Lax pair, bilinear form, Bäcklund transformation of integrable models but also exact solutions both for integrable or nonintegrable models.

In [2], Conte had proposed a simplification of the WTC approach which is corresponding to the resummation of the usual WTC approach such that the new expansion coefficients are all invariant under the Möbius transformation. The standard truncation in Conte's analysis is related to a special type of nontruncated summation in the usual WTC approach. According to Conte's analysis, a special kind of similarity reduction can be obtained [3], which can also be obtained from the CK's (Clarkson and Kruskal [4, 5]) direct method or the so-called nonclassical Lie approach [6]. In [7], Pickering proposed a nonstandard truncation approach basing on Conte's Painlevé expansion. If an original nonlinear PDE possesses more than one branch in the usual WTC expansion, then some new nontrivial exact solutions can be obtained due to Pickering's nonstandard truncation approach.

Similar to Conte's consideration, we may obtain some other types of expansions to study the Painlevé property if we relax Conte's two requirements, because the singular manifold is arbitrary. Perhaps, the different uses of the expansions may cause complexity in the study of the Painlevé analysis. However it is useful to get different new exact solutions.

In Pickering's consideration, the nonstandard truncation will yield a new nontrivial solution only for those equations which possess more than one branch in the original WTC analysis. We hope that, when using some different expansions in the study of the Painlevé analysis, some nonstandard truncation approaches may lead to some new exact solutions, no matter whether the equations possess a single or more branches in the usual Painlevé analysis.

In the next section, we discuss the general aspect of the extended Painlevé expansion. In Sect. 3, we use the Burgers equation as a simple example to restudy its Painlevé property and to show how the nonstandard truncation approach yields some new exact solutions. Applying the same idea to many significant nonlinear equations such as the KdV, modified KdV (mKdV), KP, (1+1)- and (2+1)-dimensional CDGSK (or name BKP), NLS, DS, Liouville, sG, Mikhailov-Dodd-Bullough (MDB), Kolmogoroff-Petrovsky-Piscounov (KPP), Chazy class VII, ϕ^4, KdVB, equations etc., we find a quite universal

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reduction function which is valid for various integrable and nonintegrable models. Section 4 brings a list of the universal reductions for some physically significant equations. The last section is a short summary and discussion.

2. Extend Painlevé Expansions

For a given PDE, say

$$F(t, x_1, x_2, ..., x_n, u, u_{x_i}, u_{x_{ij}}, ...) \equiv F(u) = 0, \quad (1)$$

the usual Painlevé expansion takes the form

$$u = \phi^\alpha \sum_{j=0}^{\infty} u'_j \phi^j, \quad (2)$$

where $\phi \equiv \phi(x_1, x_2, ..., x_n, t) = 0$ is an arbitrary singular manifold. Because of ϕ being arbitrary, Conte [2], choose ($x_1 \equiv x$)

$$\chi \equiv (\frac{\phi_x}{\phi} - \frac{\phi_{xx}}{2\phi_x})^{-1} \quad (3)$$

as a new expansion variable such that the coefficients u_j, in the new expansion

$$u = \chi^\alpha \sum_{j=0}^{\infty} u_j \chi^j \quad (4)$$

are invariant under the Möbius transformation

$$\phi \longrightarrow \frac{a\phi + b}{c\phi + d}, \qquad (ad \neq cb). \quad (5)$$

Differentiating (3) with respect to x and t, respectively, one gets two identities

$$\chi_x = 1 + \frac{1}{2} S\chi^2, \quad (6)$$

$$\chi_t = -C + C_x\chi - \frac{1}{2}(C_{xx} + CS)\chi^2, \quad (7)$$

where

$$S \equiv \frac{\phi_{xxx}}{\phi_x} - \frac{3}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2 \quad (8)$$

and

$$C \equiv -\frac{\phi_t}{\phi_x}, \quad (9)$$

which are the Möbius transformation invariants. The consistency condition (cross derivative) of (6) and (7) reads

$$S_t + C_{xxx} + 2C_x S + CS_x = 0. \quad (10)$$

Now let us consider the expansion (4) with (6), (7) and (10) in an alternative way. The arbitrary expansion function ϕ is changed to χ, though it should satisfy (6) and (7). The arbitraryness of the expansion function is still preserved, because the two functions S and C are included in the two equations, and there is only one constraint on the two functions. From this point of view, we may choose a different function which is given by the pair of equations (with some other functions) as new expansion variables. If the number of constraints on the functions included in the pair equation is less than the number of functions, then the arbitrarity of the new expansion variable is preserved. For instance, we may select ξ, which is related to $2N + 2$ functions S_i and Y_i by

$$\xi_x = \sum_{j=0}^{N} S_j \xi^j, \quad (11)$$

and

$$\xi_t = \sum_{j=0}^{N} Y_j \xi^j, \quad (12)$$

as a new expansion variable. It is easy to see that there are only $2N - 1$ consistent constraints

$$S_{nt} - Y_{nx} + \sum_{j=1}^{n+1} j(S_j Y_{n+1-j} - Y_j S_{n+1-j}) = 0, \quad n = 0, 1, ..., N, \quad (13)$$

$$\sum_{j=n+1-N}^{N} j(S_j Y_{n+1-j} - Y_j S_{n+1-j}) = 0, \quad n = N+1, N+2, ..., 2N-2, \quad (14)$$

among $2N + 2$ functions. So the arbitraryness of the new expansion variable ξ is preserved because at least three arbitrary functions are included in (11) and (12). However, in this general case, the relation between the usual expansion variable ϕ and the new expansion variable ξ will be only implicit except in some special cases like Conte's expansion.

Obviously, it seems not useful to simplify the procedure of the Painlevé test. However, using the new expansions, we may get some different new exact solutions because we can use some different truncations. To see this point more clearly we turn to some concrete examples.

3. Revisit on the Painlevé Test of the Burgers Equation

To be more specific, we restrict ourselves to $N = 3$ in (11) and (12) at first, i. e., the expansion variable equations read

$$\xi_x = S_0 + S_1\xi + S_2\xi^2 + S_3\xi^3, \tag{15}$$

$$\xi_t = Y_0 + Y_1\xi + Y_2\xi^2 + Y_3\xi^3, \tag{16}$$

and the constraint equations between the functions S_i and Y_i are

$$S_2Y_3 - S_3Y_2 = 0, \tag{17}$$

$$S_{3t} - Y_{3x} + 2S_3Y_1 - 2S_1Y_3 = 0, \tag{18}$$

$$S_{2t} - Y_{2x} - S_1Y_2 + S_2Y_1 - 3Y_3S_0 + 3S_3Y_0 = 0, \tag{19}$$

$$S_{1t} - Y_{1x} + 2S_2Y_0 - 2Y_2S_0 = 0, \tag{20}$$

$$S_{0t} - Y_{0x} - Y_1S_0 + S_1Y_0 = 0. \tag{21}$$

That is to say, there are only five constraint conditions for eight functions $S_0,\ S_1,\ S_2,\ S_3,\ Y_0,\ Y_1,\ Y_2,$ and Y_3. So the new expansion variable ξ can still be considered as arbitrary.

The Burgers equation

$$u_t - 2uu_x + u_{xx} = 0 \tag{22}$$

is one of the simplest integrable models. Now we use the new expansion variable ξ which is given by (11) and (12). Substituting the expansion

$$u = \xi^\alpha \sum_{j=0}^{\infty} u_j\xi^j \tag{23}$$

with (11) and (12) into the Burgers equation (22), one can easily see that

$$\alpha = -1, \qquad u_0 = -S_0 \tag{24}$$

by using the leading order analysis; the other coefficients u_j are given by

$$\begin{aligned}(j+1)(j-2)u_j &= f_j(u_k, k = 0, 1, 2, \ldots, j-1)\\ &\equiv f_j,\end{aligned} \tag{25}$$

where f_j is a quite complicated function of the $u_0,\ u_1,\ u_2,\ \ldots,\ u_{j-1}$. From (25) we know that the resonances are located at $j = -1$ and 2. The resonance at $j = -1$ corresponds to the expansion function being arbitrary. While the resonance condition, $f_2 = 0$, at $j = 2$ should be satisfied identically because the Painlevé property of the Burgers equation was known. Writing down the next two equations of (25) explicitly for $j = 1$ and $j = 2$, we have

$$\begin{aligned}&-2u_0u_{0x} + 2u_0^2S_1 + 2u_1u_0S_0 + 3u_0S_0S_1\\ &- u_0S_{0x} - u_0Y_0 - 2u_{0x}S_0 = 0\end{aligned} \tag{26}$$

and

$$\begin{aligned}&-u_0S_{1x} + u_0S_1^2 + 2u_0S_0S_2 + u_{0t} + u_{0xx}\\ &- u_0Y_1 - 2u_{0x}S_1 + 2u_0^2S_2\\ &- 2u_0u_{1x} - 2u_1u_{0x} + 2u_1u_0S_1 = 0.\end{aligned} \tag{27}$$

Substituting $u_0 = -S_0$ into (26) yields

$$u_1 = \frac{1}{2S_0}(S_{0x} - S_0S_1 + Y_0). \tag{28}$$

Now using (24) and (28), one can see that the resonance equation (27) (at $j = 2$) is simplified to

$$S_{0t} - Y_{0x} + Y_0S_1 - S_0Y_1 = 0. \tag{29}$$

Equation (29) is just the consistency condition (21). That is to say, the resonance condition $f_2 = 0$ is satisfied identically. So the Painlevé property of the Burgers equation is re-obtained in the new expansion. Now we turn to study the truncated expansion to get some new exact solutions.

From (24) (or the standard Painlevé analysis [1]), we know that the Burgers equation possesses only one branch. So from Conte's expansion one can not obtain a new exact solution by Pickering's nonstandard truncation. The present situation is quite different. From the (15) and (16) we know that the derivative operator ∂_x (or ∂_t) possesses different degrees in the negative and positive directions. In the negative direction the operator has degree one while it possesses degree two in the positive direction. In Conte's expansion, the differential operators possess the same degree (one) in both directions. So the balance conditions in the new truncated expansion are always different in the negative and positive directions, no matter if the equation possesses one or more branches in the usual Painlevé analysis.

From the leading order analysis of the Burgers equation one can easily find that the nonstandard expansion should have the form

$$u = \frac{u_0}{\xi} + u_1 + u_2\xi + u_3\xi^2 \tag{30}$$

in order to balance the effects of the nonlinearity and those of the dispersion in positive and negative directions. Substituting the nonstandard expansion (30) into the Burgers equation (22) and canceling the coefficients of ξ^J $(J = -3, -2, \ldots, 5, 6)$ yields seven further constraints:

$$-4u_3S_3(u_3 - 2S_3) = 0, \tag{31}$$

$$14u_3S_2S_3 - 6u_2u_3S_3 - 4u_3^2S_2 + 3u_2S_3^2 = 0, \tag{32}$$

$$5u_2S_2S_3 + 4u_{3x}S_3 + 12u_3S_1S_3 + 6u_3S_2^2 - 4u_1u_3S_3 - 2u_2^2S_3 - 6u_2u_3S_2 - 2u_3u_{3x} - 4u_3^2S_1 + 2u_3Y_3 + 2u_3S_{3x} = 0, \tag{33}$$

$$\begin{aligned}&-u_0S_3^2 + 4u_2S_1S_3 + 2u_2S_2^2 + u_2S_{3x} + 10u_3S_0S_3 + 10u_3S_1S_2 + 4u_{3x}S_2 + 2u_3Y_2 + u_2Y_3 + 2u_{2x}S_3\\ &- 2u_1u_2S_3 - 4u_1u_3S_2 - 2u_0u_3S_3 - 2u_3u_{2x} - 6u_3u_2S_1 - 4u_3^2S_0 - 2u_2^2S_2 - 2u_2u_{3x} + 2u_3S_{2x} = 0,\end{aligned} \tag{34}$$

$$\begin{aligned}&-4u_1u_3S_1 + 8u_3S_0S_2 - 2u_0u_3S_2 - 6u_2u_3S_0 - 2u_1u_2S_2 + u_{3t} + u_{3xx} + u_2Y_2 + 4u_{3x}S_1 + 2u_3Y_1 + 2u_{2x}S_2\\ &+ 4u_3S_1^2 - u_0S_2S_3 - 2u_1u_{3x} - 2u_2u_{2x} - 2u_2^2S_1 - 2u_3u_{1x} + u_2S_{2x} + 2u_3S_{1x} + 3u_2S_1S_2 + 3u_2S_0S_3 = 0,\end{aligned} \tag{35}$$

$$\begin{aligned}&-2u_{0x}S_2 + 2u_3S_0^2 - u_0Y_2 + u_2Y_0 + 2u_{2x}S_0 + u_{1t} + u_{1xx} - 2u_3u_0S_0 - 2u_2u_{0x} + 2u_0^2S_3\\ &- 2u_0u_{2x} - u_0S_{2x} - 2u_1u_{1x} + u_2S_{0x} + u_0S_0S_3 + u_0S_1S_2 + 2u_1u_0S_2 - 2u_1u_2S_0 + u_2S_0S_1 = 0,\end{aligned} \tag{36}$$

$$\begin{aligned}&2u_2S_0S_2 + u_{2t} + u_{2xx} + 6u_3S_0S_1 - u_0Y_3 - 2u_{0x}S_3 + u_2Y_1 + 4u_{3x}S_0 + 2u_3Y_0 + 2u_{2x}S_1 + 2u_1u_0S_3\\ &- u_0u_{3x} - 2u_0u_{3x} - 2u_0u_3S_1 - 2u_1u_2S_1 - 4u_1u_3S_0 - 2u_1u_{2x} - 2u_2u_{1x}\\ &- 2u_2^2S_0 - 2u_3u_{0x} + 2u_3S_{0x} + u_2S_{1x} + u_2S_1 = 0\end{aligned} \tag{37}$$

in addition to (17) through (21), (24) and (28) (fourteen conditions in all) for twelve functions S_i, Y_i, and u_i. It is difficult to find out all possible solutions of the overdetermined constraint equations. As in other truncated expansion approaches, we consider only the constant solutions on these constraint equations. After some simplifications, the final result can be written as

$$\xi_x = k_1(-16 + 6\xi + 9\xi^2 + \xi^3), \tag{38}$$

$$\xi_t = k_0(-16 + 6\xi + 9\xi^2 + \xi^3) \tag{39}$$

and

$$u = 16\frac{k_1}{\xi} + \left(\frac{k_0}{2k_1} - 3k_1\right) + 12k_1\xi + 2k_1\xi^2. \tag{40}$$

with two arbitrary constants k_1 and k_0. Equations (38) and (39) can be integrated implicitly:

$$\frac{(\xi - 1)^2(\xi + 8)}{(\xi + 2)^3} = c_1e^{54\eta}, \quad \eta = k_1x + k_0x. \tag{41}$$

Obviously the solution (40) with (41) can not be obtained from the other truncated expansions because the coefficients of (38) are all fixed up to a constant of proportionality and the function ξ, expressed implicitly by (4) possesses three branches (in the complex sense).

Using the same procedure for other nonlinear PDEs, we found that the reduction function (41) is quite universal for many nonlinear PDEs. We list only the final results in the next section.

4. Examples with a Common Reduction

4.1. KP and KdV Equations

The idea used in the last section can also be used in higher dimensions. The (2+1)-dimensional KP equation

$$u_{xt} - 6u_x^2 - 6uu_{xx} + u_{xxxx} + 3\sigma^2 u_{yy} = 0 \quad (42)$$

possesses also only a single branch in the usual Painlevé expansion. Applying the new expansion approach to (42) lead to the result

$$\begin{aligned} u = {} & 512\frac{k_1^2}{\xi^2} - 192\frac{{k_1}^2}{\xi} + \frac{\sigma^2 k_2^2}{2k_1^2} - 186k_1^2 + \frac{k_0}{6k_1} \\ & - 16\xi k_1^2 + 264k_1^2\xi^2 + 96k_1^2\xi^3 + 8k_1^2\xi^4, \end{aligned} \quad (43)$$

where ξ is determined by the same reduction function (41) but with

$$\eta = k_1 x + k_2 y + k_0 t. \quad (44)$$

When $k_2 = 0$ (i. e., the model is y-independent), the result (43) with (41) becomes the solution of the KdV equation.

4.2. CDGSK and/or BKP Equations

The (2+1)-dimensional CDGSK or named BKP equation

$$\begin{aligned} & u_t + 5u_x u_{xx} + 5uu_{xxx} + 5uu_y + 5wu_x \\ & + 5u_{xxy} + u_{xxxxx} - 5w_y = 0, \end{aligned} \quad (45)$$

$$u_y = w_x, \quad (46)$$

possesses the exact solution

$$\begin{aligned} u = \frac{-1}{3k_1\xi^2}\Big(& 12k_1^3(256 - 96\xi - 93\xi^2 - 8\xi^3 \\ & + 132\xi^4 + 48\xi^5 + 4\xi^6) + \xi^2 k_2\Big), \end{aligned} \quad (47)$$

$$\begin{aligned} w = \frac{1}{15k_1^2\xi^2}(& -15360k_1^3k_2 + 5760k_1^3k_2\xi \\ & + 17006112\xi^2k_1^6 + 5580\xi^2k_1^3k_2 \\ & - 3\xi^2k_1k_0 + 20\xi^2k_2^2 + 480k_1^3k_2\xi^3 \\ & - 7920k_1^3k_2\xi^4 - 2880k_1^3k_2\xi^5 - 240k_1^3k_2\xi^6) \end{aligned} \quad (48)$$

with ξ and η being given by (41) and (44) respectively. When the model is y-independent, the result becomes that of the (1+1)-dimensional CDGSK equation.

4.3. A Boussinesq Type of Equation

From the variable separation approach of the DS equation, the following Boussinesq type of equations can be obtained [8]

$$u_t + w_{xx} + ww_x - uu_x = 0, \quad (49)$$

$$w_t - u_{xx} - wu_x - uw_x = 0. \quad (50)$$

The Boussinesq type of equation system (49,50) has also the common reduction (41) and

$$u = \frac{16ik_1^2 - 6i\xi k_1^2 + \xi k_0 + i\xi c_1 k_1 + 12ik_1^2\xi^2 + 2ik_1^2\xi^3}{k_1\xi}, \quad (51)$$

$$w = \frac{-16k_1 + \xi c_1 - 12k_1\xi^2 - 2k_1\xi^3}{\xi} \quad (52)$$

with a further constant c_1.

4.4. BK System

Another Boussinesq type equation is the so-called Broer-Kaup (BK) system [9]

$$u_t + u_{xx} - 2uu_x - 2w_x = 0, \quad (53)$$

$$w_t - w_{xx} - 2wu_x - 2uw_x = 0. \quad (54)$$

The corresponding result for the BK system has the form

$$u = -\frac{1}{2k_1\xi}(32k_1^2 - \xi k_0 - 6\xi k_1^2 + 24k_1^2\xi^2 + 4k_1^2\xi^3), \tag{55}$$

$$w = -\frac{4k_1^2}{\xi^2}\left(64 - 24\xi - 84\xi^2 - 2\xi^3 + 33\xi^4 + 12\xi^5 + \xi^6\right) \tag{56}$$

with ξ being given by (41) also.

4.6. DS and NLS Equations

The reduction of the Davey-Stewartson system [10, 8]

$$iu_t + u_{xx} + u_{yy} - 4u^2v - 2uw = 0, \tag{57}$$

$$-iv_t + v_{xx} + v_{yy} - 4v^2u - 2vw = 0, \tag{58}$$

$$w_{xx} - w_{yy} + 4u_{xx}v + 8u_xv_x + 4uv_{xx} = 0, \tag{59}$$

has the form

$$u = \frac{32{k_2}^4 - 32k_1^4 + 6\xi k_1^4 - i\xi k_0k_1^2 - 6\xi k_2^4 + i\xi k_0k_2^2 + 24\xi^2k_2^4 + 24\xi^2k_1^4 + 4\xi^3k_2^4 - 4\xi^4{k_1}^4}{2b_3\xi\left(k_2^2 + k_1^2\right)}, \tag{60}$$

$$v = \frac{b_3\left(32k_2^2 + 32k_1^2 - 6\xi k_1^2 - i\xi k_0 - 6k_2^2\xi + 24k_2^2\xi^2 + 24k_1^2\xi^2 + 4k_2^2\xi^3 + 4k_1^2\xi^3\right)}{4\xi\left(k_2^2 + k_1^2\right)}, \tag{61}$$

$$\begin{aligned} w = {}& \frac{1}{4\xi^2\left(2k_1^2k_2^2 + k_1^4 + k_2^4\right)}\Big\{4096k_1^4k_2^2 + 2048k_1^6 + 2048k_1^2k_2^4 - \left(1536k_1^4k_2^2 + 768{k_1}^6 + 768k_1^2k_2^3\right)\xi \\ & - \left(128\xi^3k_1^4k_2^2 + 64k_1^6 + 64k_1^2k_2^4\right)\xi^3 + \left(228k_1^6 - 2460k_1^4k_2^2 - k_0^2k_2^2 + k_2^2k_1^2 - 5604k_1^2k_2^4 - 2916k_2^6\right)\xi^2 \\ & + \left(2112k_1^4k_2^2 + 1056k_1^6 + 1056k_1^2k_2^4\right)\xi^4 + \left(768k_1^4k_0^2 + 384k_1^6 + 384k_1^2k_2^4\right)\xi^5 + \left(64k_1^4k_2^2 + 32k_1^6 + 32k_1^2k_2^4\right)\xi^6\Big\}, \end{aligned} \tag{62}$$

with (41), (44) and b_3 being arbitrary constants. When the fields are x-independent ($k_1 = 0$) and $w = 0$, the result becomes that of the nonlinear Schrödinger equation.

4.5. KdVB Equation

As in the usual truncated expansion approach, the method proposed here can also be used to get some exact solutions of nonintegrable models. The KdV-Burgers equation

$$u_t - 6uu_x + u_{xxx} + \sigma u_{xx} = 0, \tag{63}$$

may be one of the most important physical models because of its wide applications. The KdVB equation has also the reduction (41) with

$$\begin{aligned} u = {}& 8k_1^2\xi^4 + 96k_1^2\xi^3 + \left(\frac{4}{5}\sigma k_1 + 264{k_1}^2\right)\xi^2 + \left(-\frac{51}{40}\sigma k_1 - 16k_1^2 + \frac{1}{12000}\frac{\sigma^4}{k_1}\right)\xi \\ & - 186k_1^2 - \frac{6}{5}\sigma k_1 - \frac{1}{150}\sigma^2 + \frac{1}{6}\frac{k_0}{k_1} + \left(-192k_1^2 + \frac{32}{5}\sigma k_1\right)\xi^{-1} + 512\frac{k_1^2}{\xi^2}. \end{aligned} \tag{64}$$

But now a further constraint on the parameter k_1 is required

$$k_1 = \pm\frac{\sigma}{270}. \tag{65}$$

4.7. mKdV Equation

The modified KdV (mKdV) equation

$$u_t - 6u^2u_x + u_{xxx} = 0 \tag{66}$$

possesses two branches in the usual Painlevé expansion. In addition to Pickering's nonstandard truncation solution, a further reduction can also be obtained using the present expansion. The result is

$$u = 16\frac{k_1}{\xi} - 3k_1 + 12k_1\xi + 2k_1\xi^2 \tag{67}$$

with (41) and

$$k_0 = 1458k_1^3. \tag{68}$$

4.8. $\lambda\varphi^4$ Model

The $\lambda\varphi^4$ model

$$\varphi_{tt} - \varphi_{xx} - \varphi_{yy} - \varphi_{zz} + \mu\varphi + \lambda\varphi^4 \tag{69}$$

is another important nonintegrable model in physics. When the condition

$$k_0^2 = k_1^2 + k_2^2 + k_3^2 + \frac{1}{1458}\mu \tag{70}$$

holds, the $\lambda\varphi^4$ model possesses the reduction

$$\varphi = \frac{2\sqrt{-\lambda\mu}}{27\lambda}(2\xi^2 + 12\xi - 3 + 16\frac{1}{\xi}) \tag{71}$$

with (41) but $\eta = k_1x + k_2y + k_3z + k_0t$.

4.9. Liouville, sG and MDB Models

The equation

$$\phi_{xt} + \alpha e^{\phi} + \lambda e^{-\phi} + \mu e^{-2\phi} = 0 \tag{72}$$

is the generalization of the Liouville ($\lambda = \mu = 0$,), sine-Gordon (or prefer sinh-Gordon) ($\mu = 0$) and the Mikhailov - Bullough equations ($\alpha = 0$) [11]. Taking the transformation $e^{-\phi} = u$, we have

$$uu_{xt} - u_xu_t + \alpha u^4 + \lambda u + \mu = 0. \tag{73}$$

Equation (73) possesses the same reduction function (41) when

$$u = \frac{k_0k_1}{\alpha}\left(-\frac{512}{\xi^2} + \frac{192}{\xi} + 16\xi - 264\xi^2 - 96\xi^3 - 8\xi^4\right) - \frac{3}{2}\frac{\mu\alpha^2 - 7\cdot 2^9\cdot 3^{11}k_0^3k_1^3 - 124k_0k_1\lambda\alpha}{\alpha\left(8\cdot 3^{11}k_0^2k_1^2 + \lambda\alpha\right)}, \tag{74}$$

and the parameters k_0 and k_1 are related to the model parameters $\lambda,\ \mu$, and α by

$$27\mu^2\alpha^2 + 4\lambda + 2^4\cdot 3^{12}k_0^2k_1^2\lambda^2 - 2^6\cdot 3^{18}\mu k_0^3k_1^3 = 0. \tag{75}$$

4.10. KPP Equation

In some other cases, the constants k_1 and k_0 should be all fixed. For instance, for the so-called KPP equation [12]

$$u_t - u_{xx} + 2u^3 - u = 0, \tag{76}$$

the same reduction (41) yields

$$k_0^2 = \frac{1}{1296},\quad k_1^2 = \frac{1}{5832} \tag{77}$$

for

$$u = -\frac{a_1}{16\xi}\left(-16 + 3\xi + 972\xi k_0 - 12\xi^2 - 2\xi^3\right) \tag{78}$$

with $a_1^2 = 256k_1^2$.

4.11. Chazy Class VII

For Chazy class VII [13]

$$u_{xxx} - uu_{xx} - 2u_x^2 - 2u^2u_x = 0, \tag{79}$$

its reduction is the same as that of the mKdV equation but with $\eta = k_1x$, and there is no restriction on k_1.

5. Summary and Discussion

Because the singular manifold in the usual Painlevé analysis is arbitrary, one may expand a field in many different forms. Starting from some different expansion forms, one may take different truncation procedures to get additional exact solutions. The truncated reduction in one special expansion corresponds to a special nontruncated solution in other types of expansions.

In this paper, we have introduced a simple new expansion for many known integrable and nonintegrable models like the Burgers, KdV, KP, mKdV, (1+1)- and (2+1)-dimensional CDGSK, BK, NLS, DS, Liouville, SG (ShG), MDB, KdVB, KPP, Chazy VII, and $\lambda\varphi^4$ equations, and a common reduction is obtained. Though the reduction functions for all the mentioned models are the same for the special expansion, the constraints on the parameters are different. In contrast to the usual single soliton (or solitary wave) solutions, for the Burgers, KdV, KP, CDGSK, BK, NLS, and DS equations there is no additional dispersion relation required, i. e., there are no constraints on the moment and energy parameters k_1 (and k_2 in (2+1)-dimensions) and k_0. (The usual dispersion relations will be re-found if we add the boundary conditions on the obtained solutions, say, $u(\pm\infty) = 0$ for the KdV equation). For the mKdV, SG, MDB, KdVB, and $\lambda\varphi^4$ models, further constraints are required and these constraints are also different from the dispersion relations for the usual single soliton solutions. For the KPP equation, all the constants k_1 and k_0 are fixed.

Furthermore, some more complicated nonstandard expansions can be obtained if we take $N = 4, 5, \ldots$ in (11) and (12). However we do not discuss those more complicated nonstandard truncation expansions here because of their complexity.

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